

# DIAGRAMMATIC CHARACTERISATION OF ENRICHED ABSOLUTE COLIMITS

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**ABSTRACT.** We provide a diagrammatic criterion for the existence of an absolute colimit in the context of enriched category theory.

An *absolute colimit* is one preserved by any functor; the class of absolute colimits was characterised for ordinary categories by Paré [4] and for enriched ones by Street [5]. For categories enriched over a monoidal category  $\mathcal{V}$  or bicategory  $\mathcal{W}$ , the appropriate colimits are the weighted colimits of [6], and Street's characterisation is in fact one of the class of *absolute weights*: those weights  $\varphi$  such that  $\varphi$ -weighted colimits are preserved by any functor. This is different to Paré's result, which gives a diagrammatic characterisation of when a particular cocone is absolutely colimiting. In this note, we give a result in the enriched context which is closer in spirit to Paré's than to Street's. This result is very useful in practice, but seems not to be in the literature; we set it down for future use.

## 1. The result

**1.1. BACKGROUND.** We work in the context of bicategory-enriched category theory; see [6], for example.  $\mathcal{W}$  will denote a bicategory whose homs are locally small, complete and cocomplete categories, and which is *biclosed*, meaning that for each 1-cell  $A: x \rightarrow y$  in  $\mathcal{W}$ , the composition functors  $A \otimes (-): \mathcal{W}(z, x) \rightarrow \mathcal{W}(z, y)$  and  $(-) \otimes A: \mathcal{W}(y, z) \rightarrow \mathcal{W}(x, z)$  have right adjoints  $[A, -]$  and  $\langle A, - \rangle$  respectively.

A  $\mathcal{W}$ -category  $\mathcal{A}$  comprises a set  $\text{ob } \mathcal{A}$  of objects; for each  $a \in \text{ob } \mathcal{A}$  an object  $\epsilon a \in \text{ob } \mathcal{W}$ , the *extent* of  $a$ ; for each pair of objects  $a, b$ , a hom-object  $\mathcal{C}(b, a) \in \mathcal{W}(\epsilon a, \epsilon b)$ ; and identity and composition 2-cells  $\iota: I_{\epsilon a} \rightarrow \mathcal{C}(a, a)$  and  $\mu: \mathcal{C}(c, b) \otimes \mathcal{C}(b, a) \rightarrow \mathcal{C}(c, a)$  satisfying the expected axioms. A  $\mathcal{W}$ -profunctor  $M: \mathcal{A} \rightharpoonup \mathcal{B}$  is given by objects  $M(b, a) \in \mathcal{W}(\epsilon a, \epsilon b)$  and action maps  $\mu: \mathcal{B}(b', b) \otimes M(b, a) \otimes \mathcal{A}(a, a') \rightarrow M(b', a')$  satisfying unitality and associativity axioms. A *profunctor map*  $M \rightarrow M': \mathcal{A} \rightharpoonup \mathcal{B}$  comprises maps  $M(b, a) \rightarrow M'(b, a)$  compatible with the actions by  $\mathcal{A}$  and  $\mathcal{B}$ . The identity profunctor  $\mathcal{A}: \mathcal{A} \rightharpoonup \mathcal{A}$  has components  $\mathcal{A}(b, a)$  with action given by composition in  $\mathcal{A}$ . For profunctors  $M: \mathcal{A} \rightharpoonup \mathcal{B}$  and  $N: \mathcal{B} \rightharpoonup \mathcal{C}$  with  $\mathcal{B}$  small, the tensor product  $N \otimes_{\mathcal{B}} M: \mathcal{A} \rightharpoonup \mathcal{C}$  has components given by coequalisers

$$\sum_{b, b'} N(c, b) \otimes \mathcal{B}(b, b') \otimes M(b', a) \rightrightarrows \sum_b N(c, b) \otimes M(b, a) \rightarrow (N \otimes_{\mathcal{B}} M)(c, a)$$

and actions by  $\mathcal{C}$  and  $\mathcal{A}$  inherited from  $N$  and  $M$ . Small  $\mathcal{W}$ -categories, profunctors and profunctor maps comprise a bicategory  $\mathcal{W}\text{-}\mathbf{Mod}$ . There is a full embedding  $\mathcal{W} \rightarrow \mathcal{W}\text{-}\mathbf{Mod}$  sending  $X$  to the  $\mathcal{W}$ -category  $X$  with one object  $\star$  with  $\epsilon(\star) = X$  and  $X(\star, \star) = I_X$ .

If  $\mathcal{A}$  and  $\mathcal{B}$  are  $\mathcal{W}$ -categories, then a  $\mathcal{W}$ -functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  comprises an extent-preserving assignation on objects, together with 2-cells  $\mathcal{C}(b, a) \rightarrow \mathcal{D}(Fb, Fa)$  subject to two functoriality axioms. If  $F: \mathcal{A} \rightarrow \mathcal{C}$  and  $G: \mathcal{B} \rightarrow \mathcal{C}$  are  $\mathcal{W}$ -functors then there is an induced profunctor  $\mathcal{C}(G, F): \mathcal{A} \rightarrow \mathcal{B}$  with components  $\mathcal{C}(G, F)(b, a) = \mathcal{C}(Gb, Fa)$  and action derived from the action of  $F$  and  $G$  on homs and composition in  $\mathcal{C}$ .

Given profunctors  $M: \mathcal{A} \rightarrow \mathcal{B}$ ,  $N: \mathcal{B} \rightarrow \mathcal{C}$  and  $L: \mathcal{A} \rightarrow \mathcal{C}$  with  $\mathcal{B}$  small, a profunctor map  $u: N \otimes_{\mathcal{B}} M \rightarrow L$  is said to *exhibit  $M$  as  $[N, L]$*  if every map  $f: N \otimes_{\mathcal{B}} K \rightarrow L$  is of the form  $u \circ (N \otimes_{\mathcal{B}} \bar{f})$  for a unique  $\bar{f}: K \rightarrow M$ ; while it is said to *exhibit  $N$  as  $\langle M, L \rangle$*  if every  $f: K \otimes_{\mathcal{B}} M \rightarrow L$  is of the form  $u \circ (\bar{f} \otimes_{\mathcal{B}} M)$  for a unique  $\bar{f}: K \rightarrow N$ .

Given  $\varphi: \mathcal{A} \rightarrow \mathcal{B}$  in  $\mathcal{W}\text{-}\mathbf{Mod}$  and a functor  $F: \mathcal{B} \rightarrow \mathcal{C}$ , a  $\varphi$ -weighted colimit of  $F$  is a functor  $Z: \mathcal{A} \rightarrow \mathcal{C}$  and profunctor map  $a: \varphi \rightarrow \mathcal{C}(F, Z)$  such that for each  $C \in \mathcal{C}$ , the map

$$\varphi \otimes_{\mathcal{A}} \mathcal{C}(Z, C) \xrightarrow{a \otimes_{\mathcal{A}} 1} \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, C) \xrightarrow{\mu} \mathcal{C}(F, C) \quad (1)$$

exhibits  $\mathcal{C}(Z, C)$  as  $[\varphi, \mathcal{C}(F, C)]$ . A functor  $G: \mathcal{C} \rightarrow \mathcal{D}$  preserves this colimit just when the composite  $\varphi \rightarrow \mathcal{C}(F, Z) \rightarrow \mathcal{D}(GF, GZ)$  exhibits  $GZ$  as a  $\varphi$ -weighted colimit of  $GF$ ; the colimit is *absolute* when it is preserved by all functors out of  $\mathcal{C}$ . [5] proves that  $\varphi$ -weighted colimits are absolute if and only if  $\varphi$  admits a right adjoint in  $\mathcal{W}\text{-}\mathbf{Mod}$ .

Dually, given  $\psi: \mathcal{B} \rightarrow \mathcal{A}$  in  $\mathcal{W}\text{-}\mathbf{Mod}$  and a functor  $F: \mathcal{B} \rightarrow \mathcal{C}$ , a  $\psi$ -weighted limit of  $F$  is a functor  $Z: \mathcal{A} \rightarrow \mathcal{C}$  and map  $b: \psi \rightarrow \mathcal{C}(Z, F)$  such that for each  $C \in \mathcal{C}$ , the map

$$\mathcal{C}(C, Z) \otimes_{\mathcal{A}} \psi \xrightarrow{1 \otimes_{\mathcal{A}} b} \mathcal{C}(C, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, F) \xrightarrow{\mu} \mathcal{C}(C, F)$$

exhibits  $\mathcal{C}(C, Z)$  as  $\langle \psi, \mathcal{C}(C, Z) \rangle$ . Absoluteness of limits is defined as before; now every limit weighted by  $\psi: \mathcal{B} \rightarrow \mathcal{A}$  is absolute if and only if  $\psi$  has a *left* adjoint in  $\mathcal{W}\text{-}\mathbf{Mod}$ .

**1.2. THEOREM.** *Let  $\varphi: \mathcal{A} \rightarrow \mathcal{B}$  admit the right adjoint  $\psi: \mathcal{B} \rightarrow \mathcal{A}$  in  $\mathcal{W}\text{-}\mathbf{Mod}$ , and let  $F: \mathcal{B} \rightarrow \mathcal{C}$  and  $Z: \mathcal{A} \rightarrow \mathcal{C}$  be  $\mathcal{W}$ -functors. There is a bijective correspondence between data of the following forms:*

- (a) A map  $a: \varphi \rightarrow \mathcal{C}(F, Z)$  exhibiting  $Z$  as a  $\varphi$ -weighted colimit of  $F$ ;
- (b) A map  $b: \psi \rightarrow \mathcal{C}(Z, F)$  exhibiting  $Z$  as a  $\psi$ -weighted limit of  $F$ ;
- (c) Maps  $a: \varphi \rightarrow \mathcal{C}(F, Z)$  and  $b: \psi \rightarrow \mathcal{C}(Z, F)$  such that the following two squares commute in  $\mathcal{W}\text{-}\mathbf{Mod}(\mathcal{A}, \mathcal{A})$  and  $\mathcal{W}\text{-}\mathbf{Mod}(\mathcal{B}, \mathcal{B})$ :

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta} & \psi \otimes_{\mathcal{B}} \varphi \\ \downarrow Z & & \downarrow b \otimes_{\mathcal{B}} a \\ \mathcal{C}(Z, Z) & \xleftarrow{\mu} & \mathcal{C}(Z, F) \otimes_{\mathcal{B}} \mathcal{C}(F, Z) \end{array} \quad \begin{array}{ccc} \varphi \otimes_{\mathcal{A}} \psi & \xrightarrow{\varepsilon} & \mathcal{B} \\ a \otimes_{\mathcal{A}} b \downarrow & & \downarrow F \\ \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, F) & \xrightarrow{\mu} & \mathcal{C}(F, F) \end{array} \quad (2)$$

**Proof.** Suppose first given (a); consider the composite profunctor map

$$\varphi \otimes_{\mathcal{A}} \mathcal{C}(Z, F) \xrightarrow{a \otimes_{\mathcal{A}} 1} \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, F) \xrightarrow{\mu} \mathcal{C}(F, F) \quad (3)$$

Evaluating in the second variable at any  $a \in \mathcal{A}$  yields the map (1) exhibiting  $\mathcal{C}(Z, Fa)$  as  $[\varphi, \mathcal{C}(F, Fa)]$ ; it follows easily that (3) exhibits  $\mathcal{C}(Z, F)$  as  $[\varphi, \mathcal{C}(F, F)]$ . Applying this universality to the composite  $\varepsilon \circ F: \varphi \otimes_{\mathcal{A}} \psi \rightarrow \mathcal{B} \rightarrow \mathcal{C}(F, F)$  yields a unique map  $b: \psi \rightarrow \mathcal{C}(Z, F)$  making the right square of (2) commute; we must show that the left one does too. Arguing as before shows that

$$\varphi \otimes_{\mathcal{A}} \mathcal{C}(Z, Z) \xrightarrow{a \otimes_{\mathcal{A}} 1} \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, Z) \xrightarrow{\mu} \mathcal{C}(F, Z) \quad (4)$$

exhibits  $\mathcal{C}(Z, Z)$  as  $[\varphi, \mathcal{C}(F, Z)]$ . It thus suffices to show that the left square of (2) commutes after applying the functor  $\varphi \otimes_{\mathcal{A}} (-)$  and postcomposing with (4); which follows by a short calculation using commutativity in the right square and the triangle identities.

So from the data in (a) we may obtain that in (c), and the assignation is injective, since  $b$  is uniquely determined by universality of  $a$  and commutativity on the right of (2). For surjectivity, suppose given  $a$  and  $b$  as in (c); we must show that  $a$  exhibits  $Z$  as a  $\varphi$ -weighted colimit of  $F$ , in other words, that for each  $C \in \mathcal{C}$ , the map (1) exhibits  $\mathcal{C}(Z, C)$  as  $[\varphi, \mathcal{C}(F, C)]$ , or in other words, that for each map  $f: \varphi \otimes_{\mathcal{A}} K \rightarrow \mathcal{C}(F, C)$ , there is a unique map  $\bar{f}: K \rightarrow \mathcal{C}(Z, C)$  such that  $f = \mu \circ (a \otimes_{\mathcal{A}} \bar{f}): \varphi \otimes_{\mathcal{A}} K \rightarrow \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, C) \rightarrow \mathcal{C}(F, C)$ . For existence, we let  $\bar{f}$  be the

$$K \cong \mathcal{A} \otimes_{\mathcal{A}} K \xrightarrow{\eta \otimes_{\mathcal{A}} 1} \psi \otimes_{\mathcal{B}} \varphi \otimes_{\mathcal{A}} K \xrightarrow{b \otimes_{\mathcal{B}} f} \mathcal{C}(Z, F) \otimes_{\mathcal{B}} \mathcal{C}(F, C) \xrightarrow{\mu} \mathcal{C}(Z, C); \quad (5)$$

now rewriting with the right-hand square of (2) and using the triangle identities and  $F$ 's preservation of units shows that  $f = \mu \circ (a \otimes_{\mathcal{A}} \bar{f})$ . For uniqueness, let  $g: K \rightarrow \mathcal{C}(Z, C)$  also satisfy  $f = \mu \circ (a \otimes_{\mathcal{A}} g)$ . Substituting into (5) shows that  $\bar{f}$  is the composite

$$K \cong \mathcal{A} \otimes_{\mathcal{A}} K \xrightarrow{\eta \otimes_{\mathcal{A}} 1} \psi \otimes_{\mathcal{B}} \varphi \otimes_{\mathcal{A}} K \xrightarrow{b \otimes_{\mathcal{B}} a \otimes_{\mathcal{A}} g} \mathcal{C}(Z, F) \otimes_{\mathcal{B}} \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, C) \xrightarrow{\mu} \mathcal{C}(Z, C);$$

which by rewriting with the left square of (2) and using  $Z$ 's preservation of identities is equal to  $g$ . This proves the equivalence (a)  $\Leftrightarrow$  (c); now (a)  $\Leftrightarrow$  (b) follows by duality. ■

**1.3. EXAMPLES.** We first consider examples wherein  $\mathcal{W}$  is the one-object bicategory corresponding to a monoidal category  $\mathcal{V}$ .

- Let  $\mathcal{V} = \mathbf{Set}$ , and let  $\varphi$  be the weight for splittings of idempotents. The result recovers the bijection, for an idempotent  $e: A \rightarrow A$ , between: maps  $p: A \rightarrow B$  coequalising  $e$  and  $1_A$ ; maps  $i: B \rightarrow A$  equalising  $e$  and  $1_A$ ; and pairs  $(i, p)$  with  $pi = 1_A$  and  $ip = e$ .
- Let  $\mathcal{V} = \mathbf{Set}_*$ , and let  $\varphi$  be the weight for an initial object. The result recovers the bijection in a pointed category between: initial objects; terminal objects; and objects  $X$  with  $1_X = 0_X$ .
- Let  $\mathcal{V} = \mathbf{Ab}$ , and let  $\varphi$  be the weight for binary coproducts. The result recovers the bijection, for objects  $A, B$  in a pre-additive category, between: coproduct diagrams  $i_1: A \rightarrow Z \leftarrow B: i_2$ ; product diagrams  $p_1: A \leftarrow Z \rightarrow B: p_2$ ; and tuples  $(i_1, i_2, p_1, p_2)$  such that  $p_j i_k = \delta_{jk}$  and  $i_1 p_1 + i_2 p_2 = 1_Z$ .

- Let  $\mathcal{V} = \mathbf{V}\text{-}\mathbf{Lat}$ , and let  $\varphi$  be the weight for  $J$ -fold coproducts (for  $J$  a small set). The result recovers the bijection, for objects  $(A_j : j \in J)$  in a sup-lattice enriched category, between: coproduct diagrams  $(i_j : A_j \rightarrow Z)_{j \in J}$ ; product diagrams  $(p_j : Z \rightarrow A_j)_{j \in J}$ ; and families  $(i_j)_{j \in J}$  and  $(p_j)_{j \in J}$  with  $p_j i_k = \delta_{jk}$  and  $\bigvee_j i_j p_j = 1_Z$ .
- Let  $\mathcal{V} = k\text{-}\mathbf{Vect}$  for  $k$  a field of characteristic zero, let  $G$  be a finite group, and let  $\varphi : k \rightarrowtail kG$  be the trivial right  $kG$ -module  $k$ . By Burnside's Lemma,  $\varphi$  has right adjoint  $kG \rightarrowtail k$  given by the trivial left  $kG$ -module  $k$ . Now the result recovers the bijection, for a  $G$ -representation  $A$  in a  $k$ -linear category, between: maps  $p : A \rightarrow Z$  exhibiting  $Z$  as an object of coinvariants of  $A$ ; maps  $i : Z \rightarrow A$  exhibiting  $Z$  as an object of invariants of  $A$ ; and pairs of maps  $(i, p)$  with  $pi = 1_Z$  and  $ip = \frac{1}{|G|} \sum_{g \in G} g$ .

We conclude with two examples where  $\mathcal{W}$  is a genuine bicategory.

- Let  $(\mathcal{C}, j)$  be a subcanonical site, and let  $\mathcal{W}$  denote the full sub-bicategory of  $\mathbf{Span}(\mathbf{Sh}(\mathcal{C}))^{\text{op}}$  on objects of the form  $\mathcal{C}(-, X)$ . To any prestack  $p : \mathcal{E} \rightarrow \mathcal{C}$  over  $\mathcal{C}$ , we may (as in [1]) associate a  $\mathcal{W}$ -category with objects those of  $\mathcal{E}$ , extents  $\epsilon(a) = p(a)$ , and hom-object from  $a$  to  $b$  given by the span  $\mathcal{C}(-, pa) \leftarrow \mathcal{E}(a, b) \rightarrow \mathcal{C}(-, pb)$  in  $\mathbf{Sh}(\mathcal{C})$ ; here  $\mathcal{E}(a, b)(x)$  is the set of all triples  $(f, g, \theta)$  with  $f : pa \leftarrow x \rightarrow pb : g$  in  $\mathcal{C}$  and  $\theta : f^*(a) \rightarrow g^*(b)$  in  $\mathcal{E}_x$  (note that  $\mathcal{E}(a, b)$  is a sheaf by the prestack condition).

For any cover  $(f_i : U_i \rightarrow U)_{i \in I}$  in  $\mathcal{C}$ , we have a  $\mathcal{W}$ -category  $R[f]$  with object set  $I$ , extents  $\epsilon(i) = U_i$  and hom-objects  $R[f](j, i) = \mathcal{C}(-, U_j) \leftarrow \mathcal{C}(-, U_j \times_U U_i) \rightarrow \mathcal{C}(-, U_i)$ . There is a profunctor  $\varphi : U \rightarrowtail R[f]$  with components given by the spans  $\varphi(i, \star) = \mathcal{C}(-, U_i) \leftarrow \mathcal{C}(-, U_i) \rightarrow \mathcal{C}(-, U)$ . Writing  $\psi : R[f] \rightarrowtail U$  for the reverse profunctor, it is not hard to see that  $\varphi \dashv \psi$  (in fact they are adjoint pseudoinverse).

The result now says the following. Given a prestack  $p : \mathcal{E} \rightarrow \mathcal{C}$ , a cover  $(f_i : U_i \rightarrow U)$  in  $\mathcal{C}$ , and a family of spans  $p_{ij} : a_i \leftarrow a_{ij} \rightarrow a_j : q_{ij}$  in  $\mathcal{E}$  whose legs are cartesian over the projections  $U_i \leftarrow U_i \times_U U_j \rightarrow U_j$ , there is a bijection between: cocones  $(h_i : a_i \rightarrow a)$  in  $\mathcal{E}$  over the  $f_i$ 's that are colimiting for the diagram comprised of the  $p_{ij}$ 's and  $q_{ij}$ 's; universal objects  $a \in \mathcal{E}_U$  equipped with vertical maps  $f_i^*(a) \rightarrow a_i$  fitting into double pullback squares

$$\begin{array}{ccccc} f_i^*(a) & \longleftarrow & \cdot & \longrightarrow & f_j^*(a) \\ \downarrow & & \downarrow & & \downarrow \\ a_i & \xleftarrow{p_{ij}} & a_{ij} & \xrightarrow{q_{ij}} & a_j \end{array} ;$$

and objects  $a \in \mathcal{E}_U$  equipped with a family of maps  $(h_i : a_i \rightarrow a)$  cartesian over the  $f_i$ 's. This generalises [6, Proposition 5.2(b)]<sup>1</sup>.

- Let  $\mathcal{W}$  denote the bicategory whose objects are sets, and whose hom-category  $\mathcal{W}(X, Y)$  is the category of finitary functors  $\mathbf{Set}/Y \rightarrow \mathbf{Set}/X$ ; note that  $\mathcal{W}(X, Y) \simeq$

<sup>1</sup>The proposition numbering here is taken from the TAC reprint.

$[\mathbf{Fam}(Y) \times X, \mathbf{Set}]$ , where  $\mathbf{Fam}(Y)$  has as objects, finite lists of elements of  $Y$ , and as maps  $(y_0, \dots, y_m) \rightarrow (z_0, \dots, z_n)$ , functions  $f: [m] \rightarrow [n]$  such that  $y_i = z_{f(i)}$ . To any cartesian multicategory  $M$  (i.e., a *Gentzen multicategory* in the sense of [3]) we may associate a  $\mathcal{W}$ -category  $\mathcal{M}$  whose objects of extent  $X$  are  $X$ -indexed families of objects of  $M$ , and whose hom-object between families  $(a_x)_{x \in X}$  and  $(b_y)_{y \in Y}$  is the presheaf

$$\mathcal{M}((b_y), (a_x))(y_0, \dots, y_m; x) = M(b_{y_0}, \dots, b_{y_m}; a_x)$$

in  $[\mathbf{Fam}(Y) \times X, \mathbf{Set}]$ ; reindexing along maps in  $Y$  makes use of the cartesianness of the multicategory structure. Composition and units in  $\mathcal{M}$  follow from those in  $M$ .

Given a finite set  $X = \{x_0, \dots, x_n\}$ , let  $\varphi: 1 \rightarrow X$  be the  $\mathcal{W}$ -profunctor whose unique component is the representable  $y(x_0, \dots, x_n; \star) \in [\mathbf{Fam}(X) \times 1, \mathbf{Set}]$ . This has a right adjoint  $\psi: X \rightarrow 1$  whose unique component is the presheaf  $\Sigma_{x \in X} y(\star; x) \in [\mathbf{Fam}(1) \times X, \mathbf{Set}]$ . The result now establishes a bijection, for any finite family  $(a_0, \dots, a_n)$  of objects in a cartesian multicategory  $M$ , between data of the following three forms: first, an object  $a$  and a multimap  $i \in M(a_0, \dots, a_n; a)$ , composition with which induces bijections between  $M(b_0, \dots, b_k, a, c_0, \dots, c_\ell; d)$  and  $M(b_0, \dots, b_k, a_0, \dots, a_n, c_0, \dots, c_\ell; d)$ ; second, an object  $a$  and unary maps  $p_j \in M(a; a_j)$ , composition with which establishes bijections between  $M(b_0, \dots, b_k; a)$  and  $\Pi_j M(b_0, \dots, b_k; a_j)$ ; third, an object  $a$  and maps  $i$  and  $p_j$  as above such that  $p_j \circ i = \pi_j \in M(a_0, \dots, a_n; a_j)$  and  $i \circ (p_0, \dots, p_n) = 1_a \in M(a; a)$ . This generalises [2, Proposition 3.5].

## References

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